# On the role of discrete vector fields on classifying spaces

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**Abstract**

In this work, we present a central result that plays a key role in two ongoing projects on developing algorithms for computing discrete vector fields on classifying spaces of simplicial groups. The first one provides a new proof of a conjecture of Eilenberg and Mac Lane concerning the Bar construction. The second one develops a general method to compute the effective homology of groups via group extensions.

## 1 Introduction

This work is set in a simplicial context, we refer the unfamiliar reader to May's classical book [6] for all the simplicial notions. We now present some basic definitions concerning Effective Homology and Discrete Morse Theory that will be essential throughout this work. See [10] and [5] respectively for further details on these topics.

A *reduction* between two chain complexes $C$ and $D$, denoted by $\rho : C \Rightarrow D$, is a triple $\rho = (f, g, h) : C \Rightarrow D$ where $f : C \longrightarrow D$ and $g : D \longrightarrow C$ are chain complex morphisms often called *projection* and *injection*, respectively, and $h : C \longrightarrow C_{+1}$ is a graded group morphism of degree one, called *homotopy*, such that they satisfy the following relations: $fg = \mathbf{1}_D$, $d_C h + h d_C + gf = \mathbf{1}_C$ and $fh = hg = hh = 0$.

A *discrete vector field* $V$ on a chain complex $C$ with a distinguished basis $\beta$ is a collection of pairs of cells $V = \{(x_k; y_k)\}_{k \in K}$ such that: **(i)** every $x_k$ is an element of some $\beta_n$, in which case $y_k \in \beta_{n+1}$. The degree $n$ depends on $k$ and in general is not constant; **(ii)** each component $x_k \in \beta_n$ is a *regular face* of the corresponding $y_k \in \beta_{n+1}$, i.e., the coefficient of $x_k$ in $d_{n+1}(y_k)$ is a unit; and **(iii)** each generator (cell) of $C$ appears at most once in $V$. The cells $x_j$ and $y_j$ are called respectively a *source cell* and a *target cell*. A cell $x \in \beta_n$ that does not appear in $V$ is called a *critical cell*. A discrete vector field $V$ is called *acyclic* if, for every $n \in \mathbb{N}$, a function $\lambda_n : \beta_n \to \mathbb{N}$ is provided such that the length of every path starting from $x \in \beta_n$ is bounded by $\lambda_n(x)$.

These two notions are related in the following way: an acyclic discrete vector field on a given chain complex induces a reduction of the original chain complex to another one whose generators consist of the critical cells. We refer to [11] for a detailed proof of this relation. In this way, acyclic discrete vector fields arise as essential tools in the aim of obtaining simpler chain complexes where we can obtain the homological information of the original one. Moreover, the Effective Homology techniques can be applied to get algorithmic versions of all these constructions.

The objective is to continue the work started by Romero and Sergeraert in [8] where an acyclic discrete vector field $V_{EZ}$ producing the Eilenberg–Zilber reduction [4] is constructed. We extend this result to the case where the involved simplicial sets are equipped with additional discrete vector fields. As in the previous results of [8], our new results are implemented[1] in the computer algebra system Kenzo [2]. As an

[1]See `https://github.com/Juanandete/Kenzo` for its implementation.

application, we study the classifying space of a reduced simplicial group and obtain the effective homology of a group via a group extension.

## 2 Main result

Let $X$ be a simplicial set and $V$ an acyclic discrete vector field on it. We define the *zero-face filtered discrete vector field*, that we denote by $\widetilde{V}^Z$, as the acyclic discrete vector field $\widetilde{V}^Z := V - V^Z$ where $V^Z := \{(\sigma, \tau) \in V : \sigma \in X_n \wedge \partial_0 \tau = \sigma\}$. In the same way, we define the *last-face filtered discrete vector field*, that we denote by $\widetilde{V}^L$, as the acyclic discrete vector field $\widetilde{V}^L := V - V^L$ where $V^L := \{(\sigma, \tau) \in V : \sigma \in X_n \wedge \partial_{n+1} \tau = \sigma\}$. Removing vectors from a discrete vector field preserves acyclicity. We say $V$ to be *nice over the zero face* if and only if $V = \widetilde{V}^Z$ and $V$ to be *nice over the last face* if and only if $V = \widetilde{V}^L$. We can now present the main result of this work.

**Theorem 2.1** *Let $X$ and $Y$ be two simplicial sets and $V_X$ and $V_Y$ two acyclic discrete vector fields on $X$ and $Y$, respectively. Then an acyclic discrete vector field, denoted by $V_{X\times Y}$, can be defined on $X \times Y$ as follows:*

- *For a given simplex $\sigma = (\eta' \overline{x}, \eta'' \overline{y})$, apply the $V_{EZ}$ criterion to it: if it is source or target then assign the status of source or target; otherwise, if critical, then apply the $\widetilde{V}_X^Z(\overline{x})$ criterion, if source or target then assign the status of source or target; finally, if critical, then apply the $\widetilde{V}_Y^L(\overline{y})$ criterion. We will say that a simplex is detected in the first, second or third stage if its status is detected in the $V_{EZ}$ step, the $\widetilde{V}_X^Z$ step or the $\widetilde{V}_Y^L$ step, respectively.*
- *For a given source $n$-simplex $\sigma = (\eta' \overline{x}, \eta'' \overline{y})$, we define its corresponding target simplex in the following way: if the status is assigned in the first stage of the assignment process, then $V_{X\times Y}(\sigma) := V_{EZ}(\sigma)$; if it is assigned in the second stage, then $V_{X\times Y}(\sigma) := \left(\eta_{|\overline{y}|-1} \cdots \eta_0 \widetilde{V}_X^Z(\overline{x}), \eta_n \cdots \eta_{|\overline{y}|} \overline{y}\right)$; and, finally, if it is assigned in the third stage, then $V_{X\times Y}(\sigma) := \left(\eta_{|\overline{y}|} \cdots \eta_0 \overline{x}, \eta_n \cdots \eta_{|\overline{y}|+1} \widetilde{V}_Y^L(\overline{y})\right)$.*

In fact we prove that the result just introduced is both zero-nice and last-nice, allowing the process to be repeated in an inductive way without the need of filtering the zero and last face operators subsequently. We also note that the criterion for defining this discrete vector field can be swapped. These possible choices come from variations on the definition of the Eilenberg–Zilber discrete vector field, which leads to several, but equivalent, Eilenberg–Zilber reductions.

## 3 The Bar construction reduction

In their outstanding work [3], Eilenberg and Mac Lane proved that, for a reduced simplicial group $G$, there exists a homology equivalence $H_\bullet(C(BG)) \cong H_\bullet(\mathbf{Bar}(C(G)))$ between the normalized chain complex of the classifying space of a reduced simplicial group $G$ and the corresponding Bar construction of its normalized chain complex $C(G)$. Moreover, they established the explicit quasi-isomorphism $g : \mathbf{Bar}(C(G)) \longrightarrow C(BG)$ and conjectured that it was the injection of a certain reduction. In his thesis [7], Real solved this conjecture by means of the Homological Perturbation Theory, not yet developed at the time of Eilenberg and Mac Lane. We produce another answer for this conjecture in the framework of the discrete Morse theory, leading not only to a geometrical description of Real's solution but its implementation on the Kenzo system. We now proceed to sketch our solution to the conjecture.

The classifying space of a simplicial group is equipped with a natural simplicial filtration $\widetilde{BG}^{(0)} \subset \widetilde{BG}^{(1)} \subset \cdots \subset \widetilde{BG}^{(k)} \subset \cdots \subset BG$ regarding the number of non-trivial components of the simplices. In a

similar way, one can define a filtration regarding the tensor dimension of the elements of the Bar construction[2]: $\widetilde{\mathbf{Bar}}(G)^{(0)} \subset \widetilde{\mathbf{Bar}}(G)^{(1)} \subset \cdots \subset \widetilde{\mathbf{Bar}}(G)^{(k)} \subset \cdots \subset \mathbf{Bar}(G)$. Now we consider the corresponding factors of the filtration, $BG^{(k)} := \widetilde{BG}^{(k)}/\widetilde{BG}^{(k-1)}$ and $\mathbf{Bar}(G)^{(k)} := \widetilde{\mathbf{Bar}}(G)^{(k)}/\widetilde{\mathbf{Bar}}(G)^{(k-1)}$ for $k > 0$, as done by Eilenberg and Mac Lane and later by Real. We can therefore define the following simplicial set:

$$\Sigma(\overline{G \times BG^{(k-1)}}) := \Sigma\left(G \times BG^{(k-1)}/(e_0 \times BG^{(k-1)} \cup G \times \star)\right).$$

As it can be proven that there exists a semi-simplicial isomorphism $\varphi : BG^{(k)} \longrightarrow \Sigma(\overline{G \times BG^{(k-1)}})$, we can define an acyclic discrete vector field $V_{BG}^{(k)}$ recursively on $BG^{(k)}$ in the following way: for $k = 1$, we define $V_{BG}^{(1)}$ as the trivial discrete vector field, now let us suppose that $V_{BG}^{(k-1)}$ has already been defined. For a given simplex we consider its corresponding image under the $\varphi$ morphism, now we define the $V_{BG}^{(k)}$ acyclic discrete vector field by an application of Theorem 2.1 considering the trivial discrete vector field on the $G$ component and the already defined $V_{BG}^{(k-1)}$ on the second component. Note that the discrete vector field is easily uplifted by the suspension and the corresponding quotient does not alter the discrete vector field as one can verify that the corresponding simplices are all critical. It can be checked that the corresponding critical complex at each case is isomorphic to $\mathbf{Bar}(G)^{(k)}$. We also prove that under these hypothesis, the discrete vector field $V_{BG} := \bigsqcup_{k\geq 1} V_{BG}^{(k)}$ defines an acyclic discrete vector field on $BG$. The last part of the proof consists of checking that the corresponding differential of the corresponding critical complex is equivalent to the Bar construction differential.

## 4 Effective homology of group extensions

Let $1 \to G \xrightarrow{i} H \xrightarrow{j} K \to 1$ be a *constructive group extension*, i.e., it is endowed with two *set-theoretical* maps: a *(pseudo-)section* $\sigma : K \to H$ and a *(pseudo-)retraction* $\rho : H \to G$ satisfying $\rho i = \mathbf{1}_G$, $(i\rho)\cdot(\sigma j) = \mathbf{1}_H$ and $j\sigma = \mathbf{1}_K$. Our goal is to obtain the effective homology of the middle group $H$ through the effective homologies of $G$ and $K$.

Under these circumstances, the group extension gives rise not only to a fibration but a fibre bundle between the corresponding classifying spaces, $BG \hookrightarrow BH \to BK$. In this way, we can establish an explicit twisted Cartesian product and an explicit isomorphism of simplicial sets $BH \cong BG \times_\tau BK$. Now that the classifying space of the middle group can be expressed in terms of a twisted Cartesian product we intend to apply our main result. However, a direct application is not possible as the zero-face differs from the ordinary Cartesian product. To this aim, we provide a sufficient condition for the discrete vector field to remain acyclic: if the acyclic discrete vector field acting on $BK$ is zero-nice, then the discrete vector field defined in Theorem 2.1 remains acyclic. This result fully generalizes previous work where the particular cases of central extensions [9] and split extensions [1] are studied.

## 5 Conclusions

As we have seen in the present work, acyclic discrete vector fields constitute not only a tool for performing homological computations and its easy machine implementation but a way for understanding in a more geometrical way the explicit behavior of a given reduction.

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