Factorization of Resultants for B_n-Equivariant Polynomial Systems

Gael Druez and Cordian Riener
Department of Mathematics and Statistics
UiT The Arctic University of Norway
Tromsø, Norway

Abstract

We study multivariate resultants of homogeneous polynomial systems equivariant under the hyperoctahedral group $B_n = (\pm 1)^n \rtimes S_n$, acting by sign changes and permutations of the variables. B_n-equivariance forces each component polynomial to admit the canonical form $F_i(X_1, \ldots, X_n) = X_i H_i(X_1^2, \ldots, X_n^2)$, separating sign and permutation symmetry. Exploiting this structure, we derive a universal two-level factorization of the resultant. The first level, indexed by subsets of $[n]$, arises from sign symmetry through multiplicativity and specialization. After a quadratic change of variables, each factor becomes the resultant of an S_q-equivariant system, to which the theorem of Busé–Karasoulou [1] applies. This yields an explicit product formula indexed by subsets and partitions, with computable multiplicities and exponents. Computationally, the method reduces large resultants to families of smaller symmetric resultants in at most $(d-1)/2$ variables. As an application, we obtain a factorization of the discriminant of B_n-invariant polynomials.

1 Introduction

Resultants are fundamental elimination invariants in symbolic computation and computational algebraic geometry. Their computation rapidly becomes infeasible in high dimension, and exploiting algebraic structure is essential for effective elimination theory. Symmetry is among the strongest forms of such structure. Busé and Karasoulou [1] showed that the resultant of an S_n-equivariant homogeneous polynomial system admits a canonical factorization indexed by integer partitions, with explicit multiplicities. Their result reduces a single large resultant computation to a family of lower-dimensional "prototype" resultants. Since S_n is the reflection group of type A_{n-1}, a natural question is whether analogous factorizations hold for other reflection groups. In this work, we consider systems equivariant under the hyperoctahedral group $B_n = (\pm 1)^n \rtimes S_n$, the reflection group of type B_n, acting by signed permutations.

We show that B_n-equivariance induces a two-level factorization: the reflection part $(\pm 1)^n$ gives an outer factorization indexed by subsets of $[n]$, while the permutation part S_n gives an inner factorization indexed by partitions via the Busé–Karasoulou theorem. The two levels combine into a universal closed-form product formula.

2 B_n-Equivariant Systems

Let $F = (F_1, \ldots, F_n) \in k[X_1, \ldots, X_n]^n$ be a homogeneous polynomial system over a field k of characteristic $\neq 2$. The hyperoctahedral group B_n acts on k^n by

$$(\varepsilon, \sigma) \cdot (X_1, \ldots, X_n) = (\varepsilon_1 X_{\sigma(1)}, \ldots, \varepsilon_n X_{\sigma(n)}),$$

where $\varepsilon_i \in \{\pm 1\}$ and $\sigma \in S_n$. The system F is *B_n-equivariant* if $F((\varepsilon, \sigma) \cdot X) = (\varepsilon, \sigma) \cdot F(X)$ for all $(\varepsilon, \sigma) \in B_n$.

Proposition 1 (Canonical normalization). *Let $F = (F_1, \ldots, F_n)$ be a homogeneous B_n-equivariant polynomial system of degree d. Then d is odd, and each component satisfies*

$$F_i(X_1, \ldots, X_n) = X_i H_i(X_1^2, \ldots, X_n^2),$$

where $H = (H_1, \ldots, H_n)$ is S_n-equivariant and homogeneous of degree $e = (d-1)/2$.

The proof is elementary: applying the equivariance condition with $\varepsilon_i = -1$ and all other signs positive shows that F_i is odd in X_i and even in every X_j with $j \neq i$. Factoring out X_i yields a polynomial in squared variables, and the S_n-equivariance of H follows by restricting to $\varepsilon = (1, \ldots, 1)$.

In addition to the normalization $F_i = X_i H_i$, equivariance implies divisibility properties that generalize those in the S_n case: $F_i - F_j \in \langle X_i - X_j \rangle$, $F_i + F_j \in \langle X_i + X_j \rangle$, and $F_i \in \langle X_i \rangle$ for all $i \neq j$. These three families correspond precisely to the three types of positive roots of B_n: the roots $e_i - e_j$ (shared with A_{n-1}), the roots $e_i + e_j$, and the short roots e_i.

3 Divided Differences for B_n

A key technical ingredient is an adaptation of divided differences to the B_n setting. Let $p_j = \frac{1}{2j} \sum_{i=1}^n X_i^{2j}$ denote the basic invariants of B_n, and let $I = \{i_1, \ldots, i_k\} \subseteq [n]$. The partial Jacobian $J^I(p_1, \ldots, p_k)$ equals $\left(\prod_{i \in I} X_i\right) \cdot V(X_{i_1}^2, \ldots, X_{i_k}^2)$, where V denotes the Vandermonde determinant. Replacing the last column of the partial Jacobian matrix by $F_{i_1}, \ldots, F_{i_k}$ gives a determinant D_F^I that is divisible by J^I, thanks to the three divisibility properties above. This allows us to define the *B_n-divided difference*

$$F^I := \frac{D_F^I}{J^I} \in k[X_1, \ldots, X_n].$$

These divided differences satisfy an induction formula that mirrors the classical one from [1], but with squared variables in place of ordinary ones:

$$F^{\widehat{\imath}_1} - F^{\widehat{\imath}_2} = (X_{i_2}^2 - X_{i_1}^2)\, F^I,$$

where $\widehat{\imath}_j = I \setminus \{i_j\}$. In particular, this shows inductively that all B_n-divided differences are polynomials in the squared variables $X_1^2, \ldots, X_n^2$, a fact that is essential for the reduction in Section 4.

Remark 1. *In the S_n case of [1], the first-order divided differences are the polynomials themselves: $F^{\{i\}} = F_i$. In the B_n case, the normalization shifts the starting point: $F^{\{i\}} = F_i / X_i = H_i(X_1^2, \ldots, X_n^2)$. The induction formula with squared differences then plays the same structural role as the classical one, but relative to the root system of type B_n rather than A_{n-1}.*

4 Main Factorization Theorem

The canonical normalization separates sign symmetry from permutation symmetry and leads to a two-level factorization. For a proper subset $I \subsetneq [n]$ with complement $J = \{j_1, \dots, j_q\} = [n] \setminus I$, write $G^{(I)} = (G^{(I)}_{j_1}, \dots, G^{(I)}_{j_q})$ for the S_q-equivariant system of degree e obtained from the normalized system H by specializing $X_i = 0$ for $i \in I$ and passing to the variables $Y_j = X_j^2$, $j \in J$.

Theorem 1. *Let $F = (F_1, \dots, F_n)$ be a B_n-equivariant homogeneous polynomial system of degree $d = 2e + 1$. Then*

$$\operatorname{Res}(F_1, \dots, F_n) = \prod_{I \subsetneq [n]} \left(R_I \prod_{\substack{\lambda \vdash q \\ \ell(\lambda) \le e}} \operatorname{Res}\big(G^{(I)}_{\lambda,1}, \dots, G^{(I)}_{\lambda,\ell(\lambda)}\big)^{m_\lambda} \right)^{2^{q-1}}$$

where $q = |J|$, the product over λ is the Busé–Karasoulou factorization [1] of $\operatorname{Res}(G^{(I)})$ with partition-indexed specialized divided differences $G^{(I)}_{\lambda,r}$ and multiplicities m_λ, and $R_I = 1$ if $e \ge q$ and $R_I = (G^{(I),\{j_1,\dots,j_{e+1}\}})^{m_0}$ otherwise.

The proof proceeds in three steps.

Step 1: Subset expansion. Multiplicativity of the resultant applied to $F_i = X_i H_i$ yields

$$\operatorname{Res}(F_1, \dots, F_n) = \prod_{I \subseteq [n]} \operatorname{Res}(\{X_i\}_{i \in I}, \{H_j\}_{j \notin I}).$$

The factor $I = [n]$ equals $\operatorname{Res}(X_1, \dots, X_n) = 1$ and is discarded.

Step 2: Specialization and squaring. Stability of the resultant under specialization [6] reduces each factor to a resultant in $q = |J|$ variables. Since the specialized $H^{(I)}_j$ depend only on squared variables, we pass to $G^{(I)}$ via the squaring map. Jouanolou's composition formula [6] applied to the squaring map produces the exponent 2^{q-1}.

Step 3: Symmetric factorization. Since $G^{(I)}$ is S_q-equivariant and homogeneous of degree e, the Busé–Karasoulou theorem [1] provides the partition-indexed factorization of each $\operatorname{Res}(G^{(I)})$.

We consider the following example.

Example 1. *Let $n = 2$ and $d = 7$ (so $e = 3$), and consider the B_2-equivariant system*

$$F_1 = a_0 X_1^7 + a_2 X_1^5 X_2^2 + a_4 X_1^3 X_2^4 + a_6 X_1 X_2^6, \qquad F_2 = a_6 X_1^6 X_2 + a_4 X_1^4 X_2^3 + a_2 X_1^2 X_2^5 + a_0 X_2^7.$$

Theorem 1 yields three outer factors indexed by $I = \{1\}, \{2\}, \emptyset$. The first two contribute a factor a_0^2.

For $I = \emptyset$, setting $Y_i = X_i^2$ gives the S_2-equivariant system

$$G_1 = a_0 Y_1^3 + a_2 Y_1^2 Y_2 + a_4 Y_1 Y_2^2 + a_6 Y_2^3, \qquad G_2 = a_6 Y_1^3 + a_4 Y_1^2 Y_2 + a_2 Y_1 Y_2^2 + a_0 Y_2^3.$$

The Busé–Karasoulou factorization extracts the factor $G_1(1,1) = \sum_{k=0}^{3} a_{2k}$ corresponding to $\lambda = (2)$, while the remaining factor comes from $\lambda = (1,1)$.

A direct Oscar computation gives

$$\operatorname{Res}(F_1, F_2) = a_0^2 \Big(\textstyle\sum_k a_{2k}\Big)^2 \Big(\textstyle\sum_k (-1)^k a_{2k}\Big)^2 K_3(a_0, a_2, a_4, a_6)^4,$$

where K_3 is irreducible, exactly as predicted.

5 Discriminant Application and Perspectives

If $f \in k[X_1, \dots, X_n]$ is a homogeneous polynomial invariant under B_n, then its gradient ∇f is B_n-equivariant (by the chain rule and orthogonality of signed permutation matrices). Since $\mathrm{Disc}(f) = \mathrm{Res}(\partial f/\partial X_1, \dots, \partial f/\partial X_n)$ up to a constant, Theorem 1 applies directly.

Example 2. *Let $f(X_1, X_2) = a(X_1^4 + X_2^4) + bX_1^2X_2^2$. The gradient system*

$$\nabla f = (4aX_1^3 + 2bX_1X_2^2,\ 4aX_2^3 + 2bX_1^2X_2)$$

is B_2-equivariant of degree $d = 3$. A direct computation gives

$$\mathrm{Disc}(f) = 256a^2(2a - b)^2(2a + b)^2.$$

This agrees with Theorem 1: the factor $16a^2$ arises from the subset terms $I = \{1\}$ and $I = \{2\}$, while $(4a - 2b)^2(4a + 2b)^2$ comes from the Busé–Karasoulou factorization of the S_2-equivariant system in squared variables for $I = \emptyset$. Here $4a - 2b$ corresponds to R_0, and $4a + 2b$ to the partition $\lambda = (2)$; the partition $(1, 1)$ is excluded since its length exceeds $(d - 1)/2 = 1$.

Algorithmically, for fixed degree d and increasing n, the maximal dimension of the symmetric subproblems is bounded by $(d-1)/2$, independently of n. When is fixed, this yields an exponential reduction in complexity compared to generic resultant computation. The gain is most pronounced for small d; if $d \gg n$, the partition $\lambda = (1, \dots, 1)$ remains admissible, and one factor essentially retains the complexity of the original resultant. For example, for a generic system of $n = 3$ polynomials of degree $d = 5$ in three variables, the algorithm computes the 6 factors in a few tens of seconds. Their degrees range from 1 to 3, with exponents between 3 and 12, yielding a total degree of 75. By contrast, a naive resultant computation does not terminate within an hour on the same machine. The approach naturally extends to other reflection groups, notably the type D_n Weyl group, where only even sign changes are allowed. More generally, reflection symmetry should induce a coarse decomposition of elimination invariants that can be refined using permutation symmetry, connecting to the program of [7, 3].

References

[1] L. Busé and A. Karasoulou. Resultant of an equivariant polynomial system with respect to the symmetric group. *J. Symb. Comput.*, 76:142–157, 2016.

[2] G. Druez. Factorization of the resultant of a W_{2n}-equivariant polynomial system. Master's thesis, Nantes Université, 2025.

[3] T. Friedl, C. Riener, and R. Sanyal. Reflection groups, reflection arrangements, and invariant real varieties. *Proc. Amer. Math. Soc.*, 146(3):1031–1045, 2018.

[4] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, 1994.

[5] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge Univ. Press, 1990.

[6] J.-P. Jouanolou. Le formalisme du résultant. *Adv. Math.*, 90(2):117–263, 1991.

[7] T. X. Vu. Computing critical points for algebraic systems defined by hyperoctahedral invariant polynomials. In *ISSAC 2022*, pages 167–175, 2022.