Complete Reductions for Difference Rings*

Yiman Gao, Jakob Obrovsky, Carsten Schneider
Johannes Kepler University Linz,
Research Institute for Symbolic Computation (RISC),
Altenberger Straße 69, 4040, Linz, Austria.
ymgao@risc.jku.at,Jakob.Obrovsky@risc.jku.at
Carsten.Schneider@risc.jku.at

Abstract

A complete reduction ϕ on a difference ring $\mathbb{A}$ with constant subfield C is a C-linear operator that enables one to decompose an element f of the ring as the sum of a difference and $\phi(f)$ such that f is summable in $\mathbb{A}$ if and only if $\phi(f) = 0$. We outline a complete reduction algorithmically for a tower of $R\Pi\Sigma^*$-extensions in which one can model (q-)hypergeometric products over roots of unity, the harmonic numbers and generalized versions that arise in combinatorics or particle physics. A key advantage of our algorithm is that it avoids solving any difference equations. Empirical results illustrate that the summation efficiency of our algorithm outperforms the Mathematica package `Sigma`.

1 Introduction

The telescoping problem is a central paradigm in symbolic summation. Given a sequence $f(k)$ that belongs to some domain of sequences, decide constructively whether there is a $g(k)$ in the same domain or in a suitable extension such that $f(k) = g(k+1) - g(k)$. If such a solution can be found, one obtains the identity $\sum_{k=a}^{b} f(k) = g(b+1) - g(a)$. This problem can be described in terms of difference algebra as follows. A difference ring (resp. field) $\mathbb{A}$ is a ring (resp. field) together with an automorphism σ. Given $f \in \mathbb{A}$, determine whether $f = \sigma(g) - g$ for some $g \in \mathbb{A}$. Karr [8, 9] introduced so-called $\Pi\Sigma^*$-fields covering big classes of indefinite nested sums and products, and provided an algorithm to solve the telescoping problem. Later the difference field setting has been generalized to $R\Pi\Sigma^*$-extensions [13, 14] involving algebraic objects like $(-1)^k$, and the telescoping problem was solved for two special types of such extensions, called "simple $R\Pi\Sigma^*$-extensions"and "basic $R\Pi\Sigma^*$-extensions". The above algorithms all rely on solving difference equations.

An alternative approach to the telescoping problem is to employ complete reduction techniques without solving any difference equations explicitly. Let V be a linear space and U a subspace of V. According to [10, Def. 5.67], a linear map from V to itself is called a *complete reduction* for U if it is idempotent and the kernel is equal to U. That is, $V = U \oplus \mathrm{im}(\phi)$. Now let $(\mathbb{A}, \sigma)$ be a

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difference ring with the set of constants[1] $C = \{c \in \mathbb{A} \mid \sigma(c) = c\}$; during our construction the set of constants will always form a subfield of $\mathbb{A}$. Set $\Delta(\mathbb{A}) := \{\sigma(g) - g \mid g \in \mathbb{A}\}$, which is a C-linear subspace of $\mathbb{A}$. Assume that ϕ is a complete reduction for $\Delta(\mathbb{A})$. Then an element f of $\mathbb{A}$ belongs to $\Delta(\mathbb{A})$ if and only if $\phi(f)$ is equal to 0. Recently, complete reductions have been developed for D-finite functions [2, 3, 4, 15] in order to compute telescopers efficiently. Similar reduction ideas in symbolic summation also appeared for the rational case [1, 11] and (q-)hypergeometric terms [5, 6].

Next, we define $R\Pi\Sigma^*$-extensions [13]. $(\mathbb{E}, \sigma)$ is called an *$R\Pi\Sigma^*$-extension* of a difference ring $(\mathbb{A}, \sigma)$ if $\mathbb{A} = \mathbb{A}_0 \subset \mathbb{A}_1 \subset \cdots \subset \mathbb{A}_n = \mathbb{E}$ is a tower of ring extensions with the same constant subfield C, and, for each $1 \le i \le n$, one of the following conditions holds:

(i) $\mathbb{A}_i = \mathbb{A}_{i-1}[y_i]$ is a ring extension subject to the relation $y_i^{\lambda_i} = 1$ for some integer $\lambda_i > 1$ and $\sigma(y_i)/y_i \in \mathbb{A}_{i-1}^*$ is a λ_ith root of unity (y_i is called an *R-monomial*);

(ii) $\mathbb{A}_i = \mathbb{A}_{i-1}[p_i, p_i^{-1}]$ is a Laurent polynomial ring extension with $\sigma(p_i)/p_i \in \mathbb{A}_{i-1}^*$ (p_i is called a *Π-monomial*);

(iii) $\mathbb{A}_i = \mathbb{A}_{i-1}[s_i]$ is a polynomial ring extension with $\sigma(s_i) - s_i \in \mathbb{A}_{i-1}$ (s_i is called a *Σ^*-monomial*).

If each generator of $\mathbb{E}$ is either a Π-monomial or Σ^*-monomial, we call it a *$\Pi\Sigma^*$-extension*.

The main result presented in this extended abstract is given below.

Theorem 1.1 *Let $(\mathbb{E}, \sigma)$ be an $R\Pi\Sigma^*$-extension of $(C(x), \sigma)$ with $\sigma(x) = x + 1$. Then one can*

(i) construct a complete reduction ϕ for $\Delta(\mathbb{E})$, and

(ii) develop an algorithm that, for every $f \in \mathbb{E}$, computes $g \in \mathbb{E}$ such that $f = \sigma(g) - g + \phi(f)$.

For convenience, $(g, \phi(f))$ is called a *Σ-pair of f w.r.t. ϕ*.

We note that the underlying algorithms generalize the versions in [13, 14] that can treat only the subclass of basic or simple $R\Pi\Sigma^*$-extensions; for details see the example below.

2 Outline of the complete reduction

Let $(\mathbb{E}, \sigma)$ be a difference ring and $\lambda \in \mathbb{Z}^+$. If there exist idempotent, pairwise orthogonal elements $e_0, e_1, \ldots, e_{\lambda-1}$ of $\mathbb{E}$ with $\sigma(e_u) = e_{u+1 \bmod \lambda}$ such that

$$\mathbb{E} = e_0\mathbb{E} \oplus e_1\mathbb{E} \oplus \cdots \oplus e_{\lambda-1}\mathbb{E} \tag{1}$$

and $e_u\mathbb{E}$ is an integral domain, then $(\mathbb{E}, \sigma)$ is called an *idempotent difference ring of order λ*.

Let $C(x)$ be the difference field with $\sigma(x) = x + 1$. Given an $R\Pi\Sigma^*$-tower $(\mathbb{E}, \sigma)$ over $(C(x), \sigma)$ and $f \in \mathbb{E}$, we construct a complete reduction ϕ for $\Delta(\mathbb{E})$ and compute a Σ-pair of f w.r.t. ϕ using an idempotent representation as follows.

(1) Compute an idempotent representation (1) of $\mathbb{E}$ using ideas from [16, 7, 14].

(2) Construct a $\Pi\Sigma^*$-tower $(\mathbb{H}, \sigma_0)$ over $(C(x), \sigma^\lambda)$ with $\mathbb{H} \subset \mathbb{E}$, and a difference isomorphism δ from $(e_0\mathbb{E}, \sigma^\lambda)$ onto $(\mathbb{H}, \sigma_0)$ (i.e., a ring isomorphism with $\delta(\sigma^\lambda(f)) = \sigma_0(\delta(f))$ for all $f \in e_0\mathbb{E}$).

[1] All rings and fields have characteristic 0 and $\mathbb{A}^*$ denotes the set of units of a ring $\mathbb{A}$.

(3) Let ψ be the linear function from $\mathbb{E}$ to $e_0\mathbb{E}$ by mapping f to $e_0\sum_{i=0}^{\lambda-1}\sigma^{-i}(f)$. Construct a complete reduction Φ for $\rho(\mathbb{H})$ in $\mathbb{H}$ with $\rho := \sigma_0 - \mathrm{id}$. Then $\phi := \delta^{-1}\circ\Phi\circ\delta\circ\psi$ is a complete reduction for $\Delta(\mathbb{E})$. Moreover, given $f \in \mathbb{E}$, $(\sum_{i=1}^{\lambda-1}\sum_{j=1}^{i} e_{i-j}\sigma^{-j}(f) + \sum_{i=0}^{\lambda-1} e_i\sigma^i(u), e_0 v)$ is a Σ-pair of f w.r.t. ϕ if (u, v) is a Σ-pair of $\delta(\psi(f))$ w.r.t. Φ.

Example 2.1 *We illustrate our machinery by simplifying $\sum_{j=0}^{n}\frac{((-1)^j(2-j)+j)j!}{2^j\lfloor\frac{j}{2}\rfloor!}$. First we construct an $R\Pi\Sigma^*$-tower $\mathbb{E} = \mathbb{Q}(x)[y][p_1, p_1^{-1}][p_2, p_2^{-1}][p_3, p_3^{-1}]$ over $(\mathbb{Q}(x), \sigma)$ with*

$$\sigma(y) = -y, \quad \sigma(p_1) = 2p_1, \quad \sigma(p_2) = (x+1)p_2 \quad \textit{and} \quad \sigma(p_3) = \frac{x+3-y(x-1)}{4}p_3$$

where y is an R-monomial with $y^2 = 1$ and p_1, p_2, p_3 are Π-monomials. Note that the Π-monomial p_3 cannot be handled in a simple or basic $R\Pi\Sigma^*$-extension (see [13, Def. 2.19] and [14, Def. 2.7]) and thus exceeds the class of extensions treated algorithmically so far. Obviously, y, p_1, p_2 model $(-1)^j$, 2^j, $j!$, respectively. Note that $\lfloor\frac{j}{2}\rfloor!$ can be expressed as $\prod_{k=1}^{j}\frac{1}{4}(k+2+(-1)^k(k-2))$. So p_3 models $\lfloor\frac{j}{2}\rfloor!$. Moreover, the given summand can be represented by $f = ((y(2-x)+x)p_2)/(p_1p_3) \in \mathbb{E}$.

Next, we compute a Σ-pair of f w.r.t. ϕ as given in the above outline. In step 1, we get the idempotent representation $e_0\mathbb{E}\oplus\sigma(e_0)\mathbb{E}$ of $\mathbb{E}$ with $e_0 = \frac{1-y}{2}$. In step 2, we obtain a Π-tower $(\mathbb{H}, \sigma_0)$ with $\mathbb{H} = \mathbb{Q}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}][p_3, p_3^{-1}]$ over $(\mathbb{Q}(x), \sigma^2)$, where $\sigma_0(p_1) = 4p_1$, $\sigma_0(p_2) = (x+1)(x+2)p_2$, and $\sigma_0(p_3) = \frac{1+x}{2}p_3$. Moreover, there is the difference isomorphism $\delta\colon e_0\mathbb{E} \to \mathbb{H}$ sending g to $g|_{y\to-1}$ for every $g \in e_0\mathbb{E}$. In step 3, we construct a complete reduction Φ for $\rho(\mathbb{H})$ in $\mathbb{H}$ with $\rho := \sigma_0 - \mathrm{id}$. In particular we get $\delta(\psi(f)) = \frac{1-y}{2}(f + \sigma^{-1}(f))|_{y\to-1} = 2(x^2-x+2)x^{-1}p_1^{-1}p_2p_3^{-1}$ and compute a Σ-pair $(4(x-1)x^{-1}p_1^{-1}p_2p_3^{-1}, 0)$ of $\delta(\psi(f))$ w.r.t. Φ. Thus, we conclude in step 3 that $((2+x-2y+xy)p_1^{-1}p_2p_3^{-1}, 0)$ is a Σ-pair of f w.r.t. ϕ. This is reflected by the identity

$$\sum_{j=0}^{n}\frac{((-1)^j(2-j)+j)j!}{2^j\lfloor\frac{j}{2}\rfloor!} = \frac{2(n+1)!}{2^n\lfloor\frac{n}{2}\rfloor!}.$$

3 Experiments

We have implemented our algorithm (`CR`) in the computer algebra system Mathematica 14.3 and compared it with the function `RefinedTelescoping` (`RT`) in the summation package `Sigma`[12]. The experiment was carried out on a computer with Linux, CPU 3.00 GHz, Intel Core i7-9700, 32G memory. We set $\mathbb{E} = \mathbb{Q}(x)[y][p, p^{-1}][s]$, where

$$\sigma(y) = -y, \quad \sigma(p) = \frac{2(2x+1)}{x+1}p \quad \text{and} \quad \sigma(s) = s - \frac{y}{x+1}$$

with $y^2 = 1$ which belongs to the class of basic $R\Pi\Sigma^*$-extensions that can be treated by `Sigma`. Here, y, p and s model $(-1)^n$, $\binom{2n}{n}$ and $\sum_{i=1}^{n}\frac{(-1)^i}{i}$, respectively. We randomly generated two dense polynomials $u_i, v_i \in \mathbb{Q}(x)[p, s]$ of degrees i in p, s whose coefficients are quotients of random polynomials in $\mathbb{Q}[x]$ with degrees 5. Let $p_i := u_i + yv_i$. Given $\sigma(p_i) - p_i$, compute p_i. Below is a summary of running timings (in seconds).

i	8	9	10	11	12	13	14	15	16
`RT`	96.98	161.75	254.48	393.28	603.06	885.60	1298.09	1814.17	2546.93
`CR`	2.46	3.30	4.30	5.51	6.99	7.55	9.68	11.79	14.43

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